**Example: DC Level in White Gaussian Noise**

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**Example 1:** Consider the multiple observations

\[ x_t = \theta + \epsilon_t, \quad t = 1, 2, \ldots, N, \]  

where \( \theta \) is a constant (DC level that we try to estimate) and \( \epsilon_t \) is white Gaussian noise, \( \epsilon_t \sim \mathcal{N}(0, \sigma^2) \).

We choose two different estimators, \( \hat{\theta} \) and \( \tilde{\theta} \), given by

\[
\hat{\theta} = \frac{1}{N} \sum_{t=1}^{N} x_t, \\
\tilde{\theta} = x_1.
\]

The first estimator \( \hat{\theta} \) is the sample mean and the second estimator \( \tilde{\theta} \) takes the value of the first observation \( x_1 \), so it does not make use of all the data. Our interests are:

- How close will estimators be to \( \theta \)?
- Which one of these two estimators will be better?

**Assessment:** We first show that the mean of each estimator is true value,

\[
\langle \hat{\theta} \rangle = \left( \frac{1}{N} \sum_{t=1}^{N} x_t \right) \\
= \frac{1}{N} \sum_{t=1}^{N} \langle x_t \rangle \\
= \theta, \\
\langle \tilde{\theta} \rangle = \langle x_1 \rangle \\
= \theta.
\]

Thus, on the average the estimators produce the true value.
Second, we compute the variances of estimators,

\[
\text{var} (\hat{\theta}) = \text{var} \left( \frac{1}{N} \sum_{t=1}^{N} x_t \right)
\]

\[
= \frac{1}{N^2} \sum_{t=1}^{N} \text{var} (x_t)
\]

\[
= \frac{1}{N^2} N \sigma^2
\]

\[
= \frac{\sigma^2}{N},
\]

and

\[
\text{var} (\tilde{\theta}) = \text{var} (x_1)
\]

\[
= \sigma^2.
\]

Thus one can see that \( \text{var} (\hat{\theta}) > \text{var} (\tilde{\theta}) \), which implies that \( \hat{\theta} \) is better than \( \tilde{\theta} \).

Finally, we show that \( \hat{\theta} \) is the maximum likelihood estimator (MLE). Since \( \epsilon_t \) is a white Gaussian sequence, a single factor of the likelihood is given by

\[
p(x_t; \theta) = N(\theta, \sigma^2).
\]

Thus the log-likelihood is given by

\[
\mathcal{L} = \log \left( \prod_{t=1}^{N} p(x_t; \theta) \right)
\]

\[
= \sum_{t=1}^{N} \log p(x_t; \theta)
\]

\[
= -\frac{N}{2} \log (2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^{N} (x_t - \theta)^2.
\]

The MLE \( \theta_{ml} \) is given by

\[
\theta_{ml} = \arg \max_{\theta} \mathcal{L}.
\]

It follows from \( \frac{\partial \mathcal{L}}{\partial \theta} = 0 \) that we have

\[
\theta_{ml} = \frac{1}{N} \sum_{t=1}^{N} x_t.
\]